Model order reduction with preservation of passivity, non-expansivity and Markov moments

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Abstract

A new model order reduction (MOR) technique is presented which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of Linear Matrix Inequalities (LMI's) to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation (ARE), which is known to be a more efficient approach. A new ARE and a frequency inversion technique are also presented to specifically deal with the important singular case. The preservation of Markov moments is also guaranteed by the judicious choice of a projection matrix.

Key words: Passivity, non-expansivity, positive-real lemma, bounded-real lemma, model order reduction
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1. INTRODUCTION

The use of model order reduction (MOR) aiming at obtaining compact descriptions of initially large linear state space models has become a standard component in computer-aided design methodologies for a large number of engineering and physics applications. For a good introductory textbook on MOR the reader is referred to [1]. Three MOR approaches can currently be distinguished [2]. The first approach consists of the SVD-based methods, comprising the balanced realization method [3] and Hankel norm approximation [4]. The second approach consists of the projection-based Krylov-subspace methods [5], comprising the Laguerre-SVD approach [6, 7]. The third approach consists of iterative methods combining aspects of both the SVD and Krylov methods [8]. In the excellent overview paper [2] both strengths and weaknesses of the three approaches are analyzed; e.g., the first and third approaches generally preserve stability, while the second approach is fast but does not in general guarantee stability (but see also [7]).
Passivity is an important property to satisfy because stable, but non-passive macro-models can produce unstable systems when connected to other stable, even passive, loads. It is well-known that passivity is equivalent with the positive-realness of the system transfer function. The equivalent form of passivity for a scattering matrix representation is non-expansivity or bounded-realness [9, 10]. It is well established that model reduction techniques with preservation of passivity mostly belong to the balanced truncation class [11–13] or are spectral interpolation-based methods [14–16]. In the case of projection-based Krylov methods the problem of preservation of passivity has been studied by several researchers; for an overview of existing approaches see [6, 17–21]. The problem with the Krylov-based passivity preserving methods is that they

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often assume a special descriptor state space setting that may not always be feasible [11].
In this paper, we present a new passivity-preserving and non-expansivity-preserving MOR technique, which does not require any special internal structure of the state space model. It is a projection-based method which exploits the solution of Linear Matrix Inequalities (LMI's) to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation (ARE), which is known to be a more efficient approach [22].
The paper is organized as follows. Section 2 describes the new technique and contains the proof of its passivity-preserving and non-expansivity-preserving properties. Section 3 deals with the important singular case and presents a new ARE and a frequency inversion technique specifically tailored to the singular case. Finally, Section 4 presents pertinent choices for the Krylov projection matrices in such a way that the Markov moments of the system are also preserved.

2. MAIN RESULTS

Notation : Throughout the paper X^T and X^H respectively denote the transpose and Hermitian transpose of a matrix X, and I_n denotes the identity matrix of dimension n. For two Hermitian matrices X and Y, the matrix inequalities $X > Y$ or $X \geq Y$ mean that $X - Y$ is respectively positive definite or positive semidefinite. Of course, $X < Y$ or $X \leq Y$ means $Y > X$ or $Y \geq X$. The closed right halfplane $\Re e\,[s] \geq 0$ is denoted $\mathbb{C}_+$.

2.1. Positive-real systems

For the real system with minimal realization

$$\dot{x} = Ax + Bu \tag{1a}$$
$$y = Cx + Du \tag{1b}$$

where $B \neq 0$, $C \neq 0$ are respectively $n \times p$ and $p \times n$ real matrices and $A \neq 0$ is a $n \times n$ real matrix, to be passive, it is required that the $p \times p$ transfer function

$$H(s) = C(sI_n - A)^{-1}B + D$$

is analytic in $\mathbb{C}_+$, such that

$$H(s) + H(s)^H \geq 0 \quad \forall\, s \in \mathbb{C}_+$$

It is well-known [9] that the positive-real lemma in Linear Matrix Inequality (LMI) format : $\exists\, P^T = P > 0$ such that

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} \leq 0 \tag{2}$$

guarantees the passivity of the system (1). With the additional stronger condition $D + D^T > 0$ (strict passivity at $s = \infty$), the LMI (2) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the algebraic Riccati equation (ARE)

$$A^TP + PA + (PB - C^T)W_p(PB - C^T)^T = 0 \tag{3}$$

where

$$W_p = (D + D^T)^{-1}$$

The ARE (3) is generally solved by constructing the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A - BW_pC & BW_pB^T \\ -C^TW_pC & -A^T + C^TW_pB^T \end{bmatrix} \tag{4}$$

Then the system (1) is passive, i.e., the LMI (2) is feasible, if and only if $\mathcal{H}$ has no purely imaginary eigenvalues [23].
Before tackling the main results, we need to define what is meant by a descriptor state space system. It is a more general system described by the differential equations

$$E\dot{x} = Ax + Bu \tag{5a}$$
$$y = Cx + Du \tag{5b}$$

where $E \neq 0$ is a $n \times n$ real matrix called the descriptor. In descriptor state space format the transfer function is given by

$$H(s) = C(sE - A)^{-1}B + D$$

Note that it is usually required that $sE - A$ is a regular matrix pencil, i.e., $\det(sE - A) = 0$ has a finite number of s values as solutions. In our case we will only need the simple nonsingular descriptor state space format with E nonsingular.
Next suppose $H(s)$ is passive. The following theorem provides a means to obtain a reduced model which preserves passivity.

Theorem 2.1. *Suppose the system (1) is passive and let $P = P^T > 0$ be a solution of the LMI (2). Let U be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function*

$$H_1(s) = CU(sU^TPU - U^TPAU)^{-1}U^TPB + D$$

is passive.

Proof. It is clear that $H_1(s)$ can be written as

$$H_1(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

$$\tilde{A} = (U^TPU)^{-1}U^TPAU$$
$$\tilde{C} = CU \quad \tilde{B} = (U^TPU)^{-1}U^TPB$$

Putting $\tilde{P} = U^TPU$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$\begin{aligned}\mathcal{L}_1 &= \begin{bmatrix} \tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B} - \tilde{C}^T \\ \tilde{B}^T\tilde{P} - \tilde{C} & -D - D^T \end{bmatrix} \\ &= \begin{bmatrix} U^T(A^TP + PA)U & U^T(PB - C^T) \\ (B^TP - C)U & -D - D^T \end{bmatrix}\end{aligned}$$

It is easy to show that the matrix $\mathcal{L}_1$ can be written as

$$\mathcal{L}_1 = \mathcal{E}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -D - D^T \end{bmatrix} \mathcal{E}$$

where

$$\mathcal{E} = \begin{bmatrix} U & 0_{n\times p} \\ 0_{p\times r} & I_p \end{bmatrix} \tag{6}$$

By virtue of the LMI (2) we conclude that $\mathcal{L}_1 \leq 0$ and the proof is complete.

2.2. Bounded-real systems

For the real system with minimal realization (1) to be non-expansive, it is required that the transfer function $H(s)$ is analytic in $\mathbb{C}_+$ such that

$$H(s)^H H(s) \leq I_p \quad \forall s \in \mathbb{C}_+$$

In this case (see [9]), it is well-known that the bounded-real lemma in LMI format : $\exists\, P^T = P > 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I_p \end{bmatrix} \leq 0 \tag{7}$$

guarantees the non-expansivity of the system (1). With the additional stronger product condition $D^T D < I_p$ (strict non-expansivity at $s = \infty$), the LMI (7) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the ARE

$$A^T P + PA + C^T C + (PB + C^T D) W_s (PB + C^T D)^T = 0 \tag{8}$$

where

$$W_s = (I_p - D^T D)^{-1}$$

The ARE (8) is solved by constructing the associated Hamiltonian matrix

$$\tilde{\mathcal{H}} = \begin{bmatrix} A + BW_s D^T C & BW_s B^T \\ -C^T \tilde{W}_s C & -A^T - C^T D W_s B^T \end{bmatrix} \tag{9}$$

where

$$\tilde{W}_s = (I_p - DD^T)^{-1}$$

Then the system (1) is non-expansive, i.e., the LMI (7) is feasible, if and only if $\tilde{\mathcal{H}}$ has no purely imaginary eigenvalues [23].

Suppose $H(s)$ is non-expansive. The following theorem provides a means to obtain a reduced model which preserves non-expansivity.

Theorem 2.2. *Suppose the system (1) is non-expansive and let $P = P^T > 0$ be a solution of the LMI (7). Let U be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function*

$$H_2(s) = CU(sU^T PU - U^T PAU)^{-1} U^T PB + D$$

is non-expansive.

Proof. Similar as Theorem 2.1. It is clear that $H_2(s)$ can be written as

$$H_2(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

$$\tilde{A} = (U^T PU)^{-1} U^T PAU$$
$$\tilde{C} = CU \quad \tilde{B} = (U^T PU)^{-1} U^T PB$$

Putting $\tilde{P} = U^T PU$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$\begin{aligned} \mathcal{L}_2 &= \begin{bmatrix} \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{C}^T \tilde{C} & \tilde{P}\tilde{B} + \tilde{C}^T D \\ \tilde{B}^T \tilde{P} + D^T \tilde{C} & D^T D - I_p \end{bmatrix} \\ &= \begin{bmatrix} U^T (A^T P + PA + C^T C) U & U^T (PB + C^T D) \\ (B^T P + D^T C) U & D^T D - I_p \end{bmatrix} \end{aligned}$$

It is easy to show that the matrix $\mathcal{L}_2$ can be written as

$$\mathcal{L}_2 = \mathcal{E}^T \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I_p \end{bmatrix} \mathcal{E}$$

with $\mathcal{E}$ as in (6). By virtue of the LMI (7) we conclude that $\mathcal{L}_2 \leq 0$ and the proof is complete.

2.3. Markov moment preservation

In Section 4 we will show how the projection matrix U can be chosen in order to preserve a selection of the so-called Markov moments of the system.

3. THE SINGULAR CASE

In the positive-real case the LMI (2) and the ARE (3) are equivalent only in the case $W_p > 0$ or $D + D^T > 0$. Similarly, in the bounded-real case the LMI (7) and the ARE (8) are equivalent only in the case $W_s > 0$ or $D^T D < I_p$. It is seen that the singular cases $D + D^T$ singular or $I_p - D^T D$ singular cannot easily be solved by means of ARE's (but see also [24] and [25] for that matter), since the pertinent Hamiltonian matrices are then undefined. On the other hand, LMI's are convex formulations and can always be solved by convex optimization [26], without needing ARE solvers and/or Hamiltonian matrices. However, we will show we can say more under sufficiently general conditions and still use the ARE formalism. Our approach differs considerably from the approaches in [24] and [25] in that in our method no state space transformations are needed to obtain the ARE's for the singular case. In order to concentrate solely on the positive-real case we first state the following equivalence Lemmas relating bounded-real and positive-real cases :

Lemma 3.1. *$H(s) = C(sI_n - A)^{-1}B + D$ minimal and bounded-real with A Hurwitz is equivalent with $G(s) = C_1(sI_n - A)^{-1}B + D_1$ positive-real for C_1, D_1 as constructed below.*

Proof. We have

$$I_p - H^T(-s)H(s) \geq 0 \quad \forall\, s \in \mathbb{C}_+$$

Now

$$I_p - H^T(-s)H(s) = I - D^T D - (D^T C + B^T W_c)(sI_n - A)^{-1}B - B^T(-sI_n - A^T)^{-1}(C^T D + W_c B)$$

where $W_c > 0$ is the controllability Grammian.
Hence taking $D_1 = (I - D^T D)/2$ and $C_1 = -D^T C - B^T W_c$ we see that

$$G^T(-s) + G(s) \geq 0 \quad \forall\, s \in \mathbb{C}_+$$

and the proof is complete.

Lemma 3.2. *If $H(s) = C(sI_n - A)^{-1}B + D$ is minimal and bounded-real such that $\det[I_p - H(s)] \neq 0$ for $\Re e[s] > 0$ then $G(s) = [I_p - H(s)]^{-1}[I_p + H(s)] = \check{C}(sI_n - \check{A})^{-1}\check{B} + \check{D}$ is minimal and positive-real with*

$$\begin{aligned} \check{A} &= A + B(I_p - D)^{-1}C, \quad \check{B} = \sqrt{2}B(I_p - D)^{-1} \\ \check{C} &= \sqrt{2}(I_p - D)^{-1}C, \quad \check{D} = (I_p - D)^{-1}(I_p + D) \end{aligned}$$

Conversely, if $G(s) = C(sI_n - A)^{-1}B + D$ is minimal and positive-real then $H(s) = [G(s) - I_p][G(s) + I_p]^{-1} = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ is minimal and bounded-real with

$$\begin{aligned} \hat{A} &= A - B(I_p + D)^{-1}C, \quad \hat{B} = \sqrt{2}B(I_p + D)^{-1} \\ \hat{C} &= \sqrt{2}(I_p + D)^{-1}C, \quad \hat{D} = (D - I_p)(D + I_p)^{-1} \end{aligned}$$

Proof. See [27].

In order to proceed in the singular positive-real case, we first need two more Lemmas :

Lemma 3.3. *If $D + D^T \geq 0$ and* $\text{rank}(D + D^T) = r < p$ *there exists a $p \times p$ orthogonal transformation matrix Γ such that*

$$\Gamma^T(D + D^T)\Gamma = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

where the $r \times r$ matrix R_r is symmetric positive definite. The positive-realness of $\tilde{H}(s) = \Gamma^T H(s)\Gamma$ is not affected by this transformation.

Proof. See [24]. Note that $r = 0$ corresponds to the totally singular case $D + D^T = 0$.

Lemma 3.4. *Suppose B and C are full rank. Then there exists a matrix $P = P^T > 0$ that satisfies $PB = C^T$ if and only if $CB = B^T C^T > 0$. Furthermore, in that case, all positive definite solutions of $PB = C^T$ are given by*

$$P = C^T(CB)^{-1}C + B_\perp X B_\perp^T$$

where X is an arbitrary $(n-p) \times (n-p)$ positive definite matrix and $B_\perp$ is the orthonormal null space of B.

Proof. See [28]. Note that if $\ker(B) = \{0\}$, which can happen when $p \geq n$, the only solution is $P = C^T(CB)^{-1}C$.

The next theorem provides an ARE approach for the singular positive-real case.

Theorem 3.1. *Suppose the positive-real singular system is as in Lemma 3.3, i.e.,*

$$D + D^T = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}$$

with R_r positive definite. Then provided the matrices

$$C_s B_s$$

and

$$\mathcal{R} = -(C_s A B_s)^T - C_s A B_s - (C_s B_r - B_s^T C_r^T) R_r^{-1} (B_r^T C_s^T - C_r B_s)$$

(constructively defined in the proof below) are symmetric positive definite, there is a positive definite solution P of the composite algebraic Riccati equation (constructively defined in the proof below) :

$$A^T P + PA + (PB_r - C_r^T) R_r^{-1} (PB_r - C_r^T)^T + (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0 \tag{10}$$

Proof. We start with the LMI formulation by means of the Lur'e equations [29] :

$$\begin{aligned} A^T P + PA &= -Q^T Q \\ PB - C^T &= -Q^T W \\ D + D^T &= W^T W \end{aligned}$$

Partitioning the matrices B, C, Q and W as

$$B = [B_r, B_s] \qquad C = [C_r^T, C_s^T]^T \qquad Q = [Q_r^T, Q_s^T]^T \qquad W = \begin{bmatrix} W_r & 0 \\ 0 & 0 \end{bmatrix}$$

we can reformulate the Lur'e equations as :

$$\begin{aligned} A^T P + PA &= -Q_r^T Q_r - Q_s^T Q_s && \text{(11a)} \\ PB_r - C_r^T &= -Q_r^T W_r && \text{(11b)} \\ PB_s - C_s^T &= 0 && \text{(11c)} \\ R_r &= W_r^T W_r && \text{(11d)} \end{aligned}$$

Eliminating equations (11d) and (11b) we obtain

$$\begin{aligned} A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T &= -Q_s^T Q_s && \text{(12a)} \\ PB_s - C_s^T &= 0 && \text{(12b)} \end{aligned}$$

If the aim were solely to solve equation (12b), we could utilize Lemma 3.4, but in general this will not be sufficient (except when $\ker(B_s) = \{0\}$) , since we also need to satisfy equation (12a). Anyway, a first necessary condition for the existence of a positive definite P is $C_sB_s = (C_sB_s)^T > 0$ (see also [24]). Next, if we right-multiply equation (12a) with B_s, we obtain

$$A^T C_s^T + PAB_s + (PB_r - C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s) = -Q_s^T Q_s B_s \tag{13}$$

Defining $W_s = Q_s B_s$ and left-multiplying equation (13) with B_s^T, we obtain

$$(C_s AB_s)^T + C_s AB_s + (C_s B_r - B_s^T C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s) = -W_s^T W_s \tag{14}$$

Defining

$$\begin{aligned} \mathcal{V} &= B_r^T C_s^T - C_r B_s \\ \mathcal{B} &= AB_s + B_r R_r^{-1}\mathcal{V} \\ \mathcal{C} &= -C_s A + \mathcal{V}^T R_r^{-1} C_r \\ \mathcal{R} &= -(C_s AB_s)^T - C_s AB_s - \mathcal{V}^T R_r^{-1}\mathcal{V} \end{aligned}$$

we can rewrite equations (13) and (14) as

$$\begin{aligned} P\mathcal{B} - \mathcal{C}^T &= -Q_s^T W_s \\ \mathcal{R} &= W_s^T W_s \end{aligned}$$

Assuming $\mathcal{R}$ positive definite, we can write

$$Q_s^T Q_s = (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T$$

yielding the following composite algebraic Riccati equation for P :

$$A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T + (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0$$

and the proof is complete.

Remark: in the totally singular case $D + D^T = 0$ the Riccati equation becomes

$$A^T P + PA + (P\mathcal{B} - \mathcal{C}^T)\mathcal{R}^{-1}(P\mathcal{B} - \mathcal{C}^T)^T = 0$$

with

$$\begin{aligned} \mathcal{B} &= AB \\ \mathcal{C} &= -CA \\ \mathcal{R} &= -(CAB)^T - CAB \end{aligned}$$

As a last result, which can also help finding the LMI matrix P in the singular case, we have the following :

Theorem 3.2. *Frequency inversion theorem : Let $H(s) = C(sI_n - A)^{-1}B + D$ be minimal and positive-real with A Hurwitz. Then $G(s) = \tilde{C}(sI_n - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ with*

$$\tilde{A} = A^{-1} \qquad \tilde{B} = A^{-1}B \qquad \tilde{C} = -CA^{-1} \qquad \tilde{D} = D - CA^{-1}B$$

is also positive real and admits the same P matrix as $H(s)$.

PROOF. It is straightforward to see that when A is Hurwitz, then A^{-1} is also Hurwitz and vice versa. Also, it is simple to see by substitution (see also [30]) that $G(s) = H(1/s)$. By positive-realness, $H(s)$ admits a factorization [29] :

$$H(s) + H(-s)^T = M(-s)^T M(s) \quad \forall\, s \in \mathbb{C}_+$$

Since the mapping $s \mapsto 1/s$ is one-to-one in (extended) $\mathbb{C}_+$, it follows that

$$G(s) + G(-s)^T = H(1/s) + H(-1/s)^T = M(-1/s)^T M(1/s) \quad \forall\, s \in \mathbb{C}_+$$

In other words $G(s)$ is positive-real. To prove it admits the same P as $H(s)$ we write the Lur'e equations

$$\begin{aligned} A^T P + PA &= -Q^T Q \\ PB - C^T &= -Q^T W \\ D + D^T &= W^T W \end{aligned}$$

Define $\mathcal{Q} = -QA^{-1}$ and $\mathcal{W} = W - QA^{-1}B$. It is easy to see that

$$\tilde{A}^T P + P\tilde{A} = -\mathcal{Q}^T \mathcal{Q}$$

Also

$$-\mathcal{Q}^T \mathcal{W} = A^{-T}\left[Q^T W - Q^T Q A^{-1} B\right] = P\tilde{B} - \tilde{C}^T$$

and finally

$$\tilde{D} + \tilde{D}^T = \mathcal{W}^T \mathcal{W}$$

which completes the proof.

Note that $\tilde{D} = H(0)$ and hence Theorem 3.2 maps the positive-realness problem from $s = \infty$ to $s = 0$. Of course it could be that both $H(\infty) + H(\infty)^T$ and $H(0) + H(0)^T$ are singular, in which case Theorem 3.1 or the approaches in [24] and [25] will provide solutions.

4. MARKOV MOMENT PRESERVATION

In the Section 2 we showed that passivity and non-expansivity can be preserved by introducing a full rank matrix U. In this section we will show how pertinent column-orthogonal projection matrices U can be constructed which also preserve the so-called Markov moments of the system. To see this, we first write the Laurent expansion of

$$H(s) = C(sI_n - A)^{-1}B + D = C(sP + G)^{-1}R + D$$

with $G = -PA, R = PB$, in the vicinity of $s = \infty$.
We have

$$H(s) = D + \sum_{k=0}^{\infty} (-1)^k s^{-k-1} C\Omega^k B$$

where $\Omega = -A$. This can be written as

$$H(s) = \sum_{k=-1}^{\infty} (-1)^k s^{-k-1} \mathcal{M}_k$$

The coefficients $\mathcal{M}_k = C\Omega^k B$, $k \geq 0$ and $\mathcal{M}_{-1} = -D$ are known (up to a sign) as the Markov moments of $H(s)$ at $s = \infty$. Next consider the $n \times r$ Krylov matrix ($r = pq \leq n$)

$$\mathcal{K} = \left[B, \Omega B, \Omega^2 B, \ldots, \Omega^{q-1} B\right]$$

and consider choosing an orthonormal basis for the columns of $\mathcal{K}$, which can be implemented by performing the 'thin' SVD of the Krylov matrix as $\mathcal{K} = U\Sigma V^T$, and where the $n \times r$ matrix U is column-orthogonal. Putting

$$\tilde{P} = U^T P U \quad \tilde{G} = U^T G U \quad \tilde{R} = U^T R$$
$$\tilde{C} = CU \quad \tilde{\Omega} = \tilde{P}^{-1}\tilde{G} \quad \tilde{B} = \tilde{P}^{-1}\tilde{R}$$

the new Markov moments are given by

$$\tilde{\mathcal{M}}_{-1} = \mathcal{M}_{-1} = -D \qquad \tilde{\mathcal{M}}_k = \tilde{C}\tilde{\Omega}^k\tilde{B} \quad k = 0, 1, \ldots$$

We are now in a position to prove (see also [31]) :

Theorem 4.1. *With the choice of of U as above, the Markov moments are equal up to order $q-1$, i.e., $\tilde{\mathcal{M}}_k = \mathcal{M}_k$ for $k = 0, 1, \ldots, q-1$.*

Proof. Since we have constructed an orthonormal basis for the columns of $\mathcal{K}$, we can write $\Omega^k B = UW_k$, $k = 0, \ldots, q-1$, where W_k is a $r \times p$ matrix. Note that we have $R = PB = PUW_0$ and $\tilde{R} = U^T R = U^T PUW_0 = \tilde{P}W_0$ and hence $\tilde{B} = \tilde{P}^{-1}\tilde{R} = W_0$. Next consider the $n \times n$ matrix

$$Z = U\tilde{P}^{-1}U^T G$$

By induction, it is easy to prove that $Z^k U = U\tilde{\Omega}^k$ for $k = 0, \ldots, q-1$ and hence

$$\tilde{\mathcal{M}}_k = \tilde{C}\tilde{\Omega}^k\tilde{B} = CZ^k UW_0 = CZ^k B \quad k = 0, \ldots, q-1$$

There remains to prove that $Z^k B = \Omega^k B$ for $k = 0, \ldots, q-1$. This is clearly the case for $k = 0$. Next suppose that $Z^k B = \Omega^k B$ for some k. Then

$$P^{-1}GZ^k B = \Omega^{k+1}B = UW_{k+1}$$

Pre-multiplying by $U^T P$ yields

$$U^T GZ^k B = U^T PUW_{k+1} = \tilde{P}W_{k+1}$$

or

$$W_{k+1} = \tilde{P}^{-1}U^T GZ^k B$$

and hence

$$Z^{k+1}B = U\tilde{P}^{-1}U^T GZ^k B = UW_{k+1} = \Omega^{k+1}B$$

which completes the proof.

Recall that by Theorems 2.1 and 2.2, the reduced order model is passive resp. non-expansive, when the original transfer function $H(s)$ is passive resp. non-expansive. Also, one often wishes to have equal Markov moments calculated about another point than infinity, or else to have Markov moments which are coefficients of a Laguerre expansion [6, 7]. All these possibilities can be dealt with by transforming the Laplace variable s by means of a real Möbius transformation

$$s = \frac{\alpha u + \beta}{\gamma u + \delta} \qquad \alpha\delta - \beta\gamma \neq 0 \tag{15}$$

The resulting transfer function in the $u-$domain is

$$(\gamma u + \delta)C\left[u(\alpha P + \gamma G) + (\beta P + \delta G)\right]^{-1} R + D$$

Now assuming that $\alpha P + \gamma G$ is nonsingular, we can define the matrices

$$\hat{B} = (\alpha P + \gamma G)^{-1}R \qquad \hat{\Omega} = (\alpha P + \gamma G)^{-1}(\beta P + \delta G)$$

After construction of a base $\hat{U}$ of the Krylov matrix

$$\hat{\mathcal{K}} = \left[\hat{B}, \hat{\Omega}\hat{B}, \hat{\Omega}^2\hat{B}, \ldots, \hat{\Omega}^{q-1}\hat{B}\right] = \hat{U}\hat{\Sigma}\hat{V}^T$$

the reduced matrices are now

$$\tilde{P} = \hat{U}^T P \hat{U} \quad \tilde{G} = \hat{U}^T G \hat{U} \quad \tilde{R} = \hat{U}^T R \quad \tilde{C} = C\hat{U}$$

For example, inserting $\alpha = s_0, \beta = \gamma = 1, \delta = 0$ in (15), we in fact perform a Taylor expansion about s_0, as in [32], and inserting $\beta = \alpha, \gamma = -1, \delta = 1$ in (15), boils down to a scaled Laguerre expansion with scaling factor $\alpha > 0$, as in [6, 7]. Of course, by Theorems 2.1 and 2.2, passivity and non-expansivity are always maintained.

5. CONCLUSION

We have presented a new model order reduction technique which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of Linear Matrix Inequalities to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation, which is known to be a faster approach. A new ARE and a frequency inversion technique are presented to specifically deal with the difficult singular case. Last but not least, we have shown how the pertinent column-orthogonal projection matrix can be constructed such that the Markov moments of the system are also preserved.

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